The multiple-scattering series (MSS) in few-nucleon systems

Vadim Baru

Institut für Theoretische Physik II, Ruhr-Universität Bochum  Germany
Institute for Theoretical and Experimental Physics, Moscow, Russia

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in collaboration with
E. Epelbaum, C. Hanhart, M. Hoferichter, A. E. Kudryavtsev, and D.R. Phillips

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**MSS: definition**

A particular class of diagrams where a meson (e.g. pion) scatters many times between a pair of nucleons.

**Mesonic atoms** – prime source of information about MN scattering lengths

**High-precision physics** ⇒ control over theor. uncertainty ⇒ EFT (ChPT)
Example: pionic atoms

\[ \pi H, \pi D, \pi^3 \text{He} \] – tool to extract \( \pi N \) scattering lengths

\[ f_{\pi N} = a^+ \delta^{ab} + a^- i\epsilon^{bac} \tau^c \] – isospin symmetric \( \pi N \) amplitude at threshold

\( a^+, a^- \)

\[ g_{\pi NN} \]

\( \pi N \sigma \)-term

NN sector, nuclear physics, pion photoproduction

strange content of the nucleon

LO in ChPT (Weinberg (1966)):

\[ a^+ = 0 \]

\[ a^- = \frac{M_\pi}{(1 + M_\pi/m_N)8\pi F_\pi^2} \]
Example: pionic atoms

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$$a^- = \frac{M_\pi}{(1 + M_\pi/m_N)8\pi F_\pi^2}$$

$\Rightarrow$ Reliable extraction of $a^+$ and $a^-$ is possible from a combined analysis of $\pi H$ and $\pi D$ data: $a^+ = (7.6 \pm 3.1) \cdot 10^{-3} M_\pi^{-1}$, $a^- = (86.1 \pm 0.9) \cdot 10^{-3} M_\pi^{-1}$

$\Rightarrow$ MSS – key contribution to $\pi D$ scattering (Brückner 1953)
Hadronic atoms

- Driving force - static Coulomb potential
- Correction due to strong interaction: \( \epsilon_{1s} = E_{1s}^{\text{exp}} - E_{1s}^{\text{Coul}} \neq 0 \)

\[
\epsilon_{1s} \sim < \Psi_p^{\text{Coul}} | V_{\pi N}^{\text{str}} | \Psi_q^{\text{Coul}} > \sim a_{\pi-p} \Psi_{\text{Coul}}(r = 0)^2
\]

(Deser et al. (1954))

- finite lifetime \( \Rightarrow \) \( \Gamma_{1s} \sim a_{\pi-p \rightarrow \pi^0 n}^2 \)

- High-precision data exist \( (\text{PSI, Gotta et al. 2008, Strauch et al. 2010}) \):
  - \( \pi H : \) \( \epsilon_{1s} = (-7.120 \pm 0.012) \text{ eV}, \) \( \Gamma_{1s} = (0.823 \pm 0.019) \text{ eV} \)
  - \( \pi D : \) \( \epsilon_{1s}^d = (2.356 \pm 0.031) \text{ eV} \) \( \Rightarrow \) \( \text{Re} a_{\pi d} \)

- High-accuracy ChPT calculation: 5% uncertainty \( (\text{VB et al. 2011}) \):

\[
\text{Re} a_{\pi d} = 2 \frac{1 + M_{\pi}/m_N}{1 + M_{\pi}/2m_N} a^+ + a_{(3\text{body})}(a^-) + a_{IV}
\]

- one-body term
- few-body effects
- isospin violation
Hierarchy of 3-body operators in ChPT: \(O(p) \sim m_\pi / m_N\)

**LO=O(1)**

The diagrams of the LO-type but with subleading vertices cancel altogether

S. Weinberg (1992), S. Beane et al. (1998)

**NLO=O(p)**

Effect of nucleon recoil in the LO diagrams

S. Beane et al. (2002)

S. Liebig et al. (2010)

V. Baru et al. (2004, 2009)

**N^{3/2}LO=O(p^{3/2})**

Effect of nucleon recoil in the LO diagrams

V. Lensky et al. (2007)

V. Baru et al. (2008)

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**N^2LO=O(p^2)**

Theor. uncertainty estimate: \((M_\pi / m_N)^2 \cdot a^{(LO)}_{\pi d} \sim 5\% a^{(exp)}_{\pi d}\)
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MSS in perturbation theory

- Double scattering term \((Q = p - p')\)

\[
A^{(2)}(Q) = \frac{a^2}{Q^2},
\]

- leading few-body term in \(\pi D\)

- Triple scattering (leading effect from nucleon pole):

\[
A^{(3)}(Q) = -4\pi a^3 \int \frac{d^3l}{(2\pi)^3} \frac{1}{l^2(1 - Q)^2} = -\frac{a^3}{2\pi|Q|} J_0
\]

\[
J_0 = \int_0^\infty \frac{dx}{x} \log \left(\frac{x + 1}{x - 1}\right)^2
\]

- Power counting:

\[
\frac{A^{(3)}}{A^{(2)}} = \frac{aQ}{2\pi} \sim \left(\frac{M_{\pi}}{4\pi F_\pi}\right)^2 J_0 \quad \Rightarrow \quad \text{NDA:} \quad J_0 \sim 1
\]

Calculation: \(J_0 \sim \pi^2\)

formally \(N^2\text{LO} \approx 5\%) \text{ but numerically } \approx 12\%

- enhancement is due to presence of Coulombic-type propagators (no mass scale)!

S.Liebig, VB, F. Ballout, C. Hanhart, A. Nogga (2011)

- Are the higher-order MSS terms also enhanced?
MSS in perturbation theory cont’d

- Quadrupole scattering:

\[ A^{(4)}(Q) = (4\pi)^2 a^4 \int \frac{d^3 l_1}{(2\pi)^3} \frac{d^3 l_2}{(2\pi)^3} \frac{1}{l_1^2(l_1 - l_2)^2(l_2 - Q)^2} \]

  - **dimensionless**: only one scale \( Q \) ⇒ integral must be a constant

  - **UV divergent** ⇒ regularization ⇒ renormalization ⇒ contact term \( f_0(\mu) \)

\[ A^{(4)}(Q) = -a^4 \log \frac{Q}{\mu} + \frac{f_0(\mu)}{32\pi^2} \]

- \( A^{(4)} \) vs. \( A^{(2)} \)

\[ \frac{A^{(4)}}{A^{(2)}} = a^2 Q^2 \sim 4\pi^2 \left( \frac{M_\pi}{4\pi F_\pi} \right)^4 \]

  - \( \pi^2 \) enhanced again

- ⇒ Problem: \( f_0(\mu) \) also \( \pi^2 \) enhanced

- No problem for \( \pi D \) scattering: \( \pi^2 N^4\text{LO} \ll N^2\text{LO} \) but relevant for KD scattering
MSS resummation

- Integral (Faddeev-type) Eq. for MSS with static nucleons

\[ T(p', p) = t_{\pi N}(0, 0)(2\pi)^3 \delta(3)(p' - p) + \int \frac{d^3p''}{(2\pi)^3} t_{\pi N}(p - p'', 0) \frac{1}{(p - p'')^2} T(p', p'') \]

- Translational invariant: \( T(p', p) = (4\pi)^2 A(Q) \)

\[ A(Q) = t_{\pi N}(0)(2\pi)^3 \frac{\pi}{2} \delta(3)(Q) + \int \frac{d^3p''}{(2\pi)^3} t_{\pi N}(p - p'') \frac{1}{(p - p'')^2} A(p' - p'') \]

- \( \pi N \) interaction has some range \( \Lambda_{\pi N} \):

\[ t_{\pi N}(p) = -4\pi a \hat{g}\left(\frac{|p|}{\Lambda_{\pi N}}\right), \quad \hat{g}(x) \to 1 \quad \text{when} \quad x \to 0 \]

- EFT is valid when \( Q < \Lambda \) (\( \sim 4\pi F_\pi \) in ChPT); \( \Lambda_{\pi N} \approx \Lambda \)

- Fourier transform of the integral Eq. \( \Rightarrow \) analytic result in \( r \)-space:

\[ A(r) = -\frac{a}{4\pi} - \frac{a}{r} g(r) A(r) \quad \Rightarrow \quad A(r) = -\frac{ar}{4\pi(r + ag(r))} \]
MSS resummation

- Resummed MSS with static nucleons:
  \[ A(r) = -\frac{ar}{4\pi(r + ag(r))} \]

- Perturbation series:
  - valid if the scattering length \( a \) is natural: \( a \lesssim \Lambda^{-1} \approx (4\pi F_\pi)^{-1} \Rightarrow \frac{a}{r} \sim \frac{Q}{\Lambda} \ll 1 \)
  - Expansion in \( a/r \) and Fourier transform reproduces the individual MSS terms

For example:
\[ A^{(4)}(Q) = -a^4 \log \frac{Q}{\Lambda} \] diverges when \( \Lambda \to \infty \)
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- The resummed MSS is much less \( \Lambda \) dependent (finite when \( \Lambda \to \infty \)):
  \[ A(r) = -\frac{ar}{4\pi(r + a)} \]

- Does the limit \( \Lambda \to \infty \) affects the physics?
  \[ \Delta A \equiv A_\Lambda(r) - A_{\Lambda \to \infty}(r), \quad \frac{<\Delta A>}{a\pi d} \leq 3\% \Rightarrow \text{less than CT at } N^2\text{LO} \]
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Conclusion: No enhanced CT is needed in the resummed MSS
Pole in the resummed MSS

\[ A(r) = -\frac{ar}{4\pi(r + ag(r))} \]

exhibits a pole if \( a < 0 \)

- natural \( a \) \( \Rightarrow \) pole is near the origin: \( r < \Lambda^{-1} \Rightarrow Q > \Lambda \Rightarrow \) beyond applicability of EFT

- unnaturally large \( a \): \( a\Lambda > 1 \) \( \Rightarrow \) pole in physical region \( r \sim Q^{-1} \sim a > \Lambda^{-1} \)

- But if \( a > \Lambda^{-1} \) \( \Rightarrow \) shallow meson\( N \) state: \( t = -\frac{4\pi}{1/a - iQ} \) \( \Rightarrow \) Efimov physics

- Unitarity and recoil corrections shift the pole from phys. region towards origin

- But for \( a \gg \Lambda^{-1} \) the pole still affects the results \( \Rightarrow \) may be resum range corrections?!
Conclusions

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• Unphysical pole in the resummed MSS with isoscalar interactions
  ➔ outside the range of applicability of EFT for natural sc. lengths
  ➔ sc. length approximation is not justified for unnatural sc. lengths
  ➔ inclusion of unitarity, recoil and m.b. range corrections is necessary
  ➔ no pole appears for isovector dominated or absorptive interactions
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- Both perturbative and resummed MSS schemes are applicable for \( \pi D \) scatt.
- Resummed MSS is justified for \( KD \) scattering length